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GUGGENHEIM AERONAUTICAL LABORATORY

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HIGHER ORDER APPROXIMATE SOLUTIONS FOR THE FLOW IN AXIAL TURBOMACHINES

Thesis by

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PASADENA, CALIFORNIA

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ABSTRACT

The theory of the three-dimensional rotational flow of an incompressible and inviscid fluid through an axial turbomachine is described and the hydrodynamical equations are simplified by considering an infinite number of blades in each row. The forces of the blades on the fluid are treated as non-conservative body forces distributed uniformly about the axis.

Formulation of the mathematical problem leads to one non-linear partial differential equation and two integral equations for the three velocity components. A linearized solution of these simultaneous equations for any prescribed blade loading is based on the consideration that the vorticity generated by the blades is transported downstream by the mean axial velocity. An iteration process which leads to solutions of greater accuracy is developed by considering for each iteration that the vorticity is transported by the velocities found by the previous iteration.

The Bessel's functions which occur in the Green's function solution are replaced by their asymptotic values and the infinite series is summed to express the solution in closed form. The iteration process is then adapted to mechanical calculations by dividing the region of vorticity into small rings of rectangular cross-section and determining the influence on the velocity of a unit change of vorticity in each of these rings. Once this influence is established it is relatively easy to calculate the velocities in any axial flow machine with any prescribed blade loading.

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INTRODUCTION

The development of turbomachinery until recent years has resulted primarily from empirical methods, both in design and analysis. Recently theoretical methods have led to the advancement of axial flow compressors and turbines even though, because of the complicated mathematical problem, many simplifying assumptions were necessarily made and solutions were approximate.

In analogy with the three dimensional wing theory, problems of flow in turbomachines may be classified as:

(1) The Direct Problem: The direct problem of calculating the flow in turbomachinery is that of determining the velocity field, the blade forces, and the distribution of energy in the fluid when the blade shape, the blade speed, and the appropriate boundary conditions are prescribed.

(2) The Inverse Problem: The inverse problem of calculating the flow in turbomachinery is that of determining the velocity field, the blade shape, and the distribution of energy in the fluid when the blade loading, the blade speed, and the boundary conditions are prescribed.

(3) The Intermediate Problem: An intermediate problem which appears to be of interest is that of determining the velocity field, the magnitude of the remaining forces, and the distribution of energy in the fluid when one blade force is prescribed and the blade shape is partially prescribed.

The direct problem arises when it is desired to investigate a given machine "off the design point".

The inverse problem may be solved for the initial design but usually, because of the structural limitation on blade shape (see appendix), it will be more practical to formulate and solve the intermediate problem.

In order to simplify the mathematical problem the fluid will be assumed to be inviscid and incompressible and the blade forces will be treated as body forces, uniformly distributed through the fluid, so that the flow is symmetrical about the axis of rotation and the vorticity is no longer shed in sheets behind each blade but is continuously distributed over the region downstream of the blade row.

An axial flow machine for which the inner and outer boundaries consist of concentric circular cylinders extending to infinity in the direction of the flow will be considered.

The mathematical problem is formulated by considering the time rate of change, along a streamsurface, of the tangential component of the vorticity vector. The difficulty of this problem lies in the solution of the non-linear partial differential equations that describe rotational fluid motion. In order to overcome this difficulty a method of iteration is developed whereby solutions of any required degree of accuracy may be obtained.

The first step of this iteration process provides a linearized solution (cf. Marble[1]) based on the assumption, analogous to the Prandtl three dimensional wing theory, that the vorticity is "transported" downstream by the mean axial velocity and is not influenced

by its own induced velocities. This linearized solution is not as accurate as might be expected from the above analogy, one reason being that the vorticity is shed in three dimensional space instead of in a two dimensional sheet so that the induced velocities are likely to be very large.

For the second step of the iteration process the vorticity is considered transported by the velocities found by the linearized solution of the first step. The succeeding approximations are obtained in the same way, in each case using the velocities of the preceding approximation.

The solution is obtained by finding the appropriate Green's function - that is, a function G $(r, z; \alpha, \beta)$ which gives the velocity, consistent with the boundary conditions, induced at any point of a circle r, z by a unit change in tangential vorticity of a vortex ring at radius α at an axial coordinate β. The Bessel functions which arise in this solution are replaced by their asymptotic values and the Green's function, which would occur as an infinite sum of Bessel functions, is by this means expressed in closed form.

The resulting expression for the radial velocity, a double integral, is modified and adopted for mechanical calculation.

I. Notation and Symbols

The flow is described (Fig. 1) in a cylindrical coordinate system r, ϑ, z, by the velocity components u, v, w, respectively. The corresponding radial, tangential, and axial velocity components are

$$\xi = -\frac{\partial v}{\partial z}$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$

$$\zeta = \frac{1}{r}\frac{\partial}{\partial r}(rv)$$

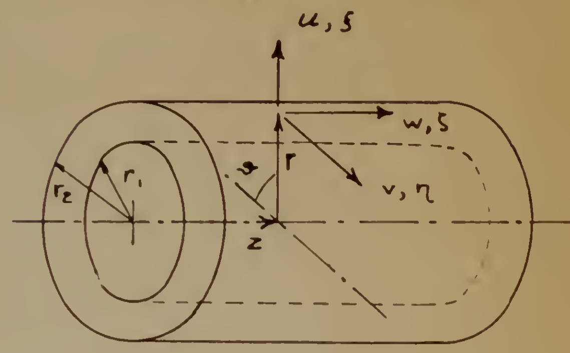

Fig. 1

Coordinate System and Designation of Velocity and Vorticity Components

In addition we will use the following symbols.

ω = angular velocity of rotor

p = pressure

$\overline{V}$ = absolute velocity vector

$\overline{\Omega}$ = vorticity vector

$\overline{F}$ = force vector (force of blades on fluid)

II. Hydrodynamic Equations

The following equations are simplified for this case of an inviscid and incompressible fluid in steady, adiabatic, axially symmetrical flow.

Equations of Motion:

$$\bar{V} \cdot \nabla \bar{V} = -\frac{1}{\rho} \nabla p + \bar{F} \tag{1}$$

$$u \frac{\partial u}{\partial r} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + F_r$$

$$\frac{u}{r} \frac{\partial}{\partial r} r v + w \frac{\partial v}{\partial z} = F_\theta$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + F_z$$

$$\bar{V} \times \bar{\Omega} = \frac{1}{\rho} \nabla p + \frac{1}{2} \nabla \bar{V} \cdot \bar{V} - \bar{F} \tag{2}$$

Continuity Equation:

$$\nabla \cdot \bar{V} = 0 \tag{3}$$

$$\frac{1}{r} \frac{\partial r u}{\partial r} + \frac{\partial w}{\partial z} = 0$$

III. Formulation of the Mathematical Problem

a) Development of the Basic Equations

Because of the axial symmetry only the tangential vorticity is associated with the radial and axial velocities, while the radial and axial vorticity components are only associated with the tangential velocity. The tangential vorticity constitutes an annular vortex ring and by considering the deformation of this ring information concerning the radial and axial velocities can be gained. It is well known (cf. Meyer[2], Marble[1]) that the circulation about a deforming vortex ring is constant in a conservative force field. Only a radial stretching can occur since the flow is symmetrical about the axis so that the constancy of circulation requires that in a conservative field

$$\frac{d}{dt}\left(\frac{\Omega}{r}\right) = 0 \qquad (4)$$

In the presence of non-conservative blade forces or if radial vorticity is present the circulation, and hence the quantity $\frac{\Omega}{r}$, will not remain constant. The law governing its variation can be derived from the equations of motion. The equations of motion in vector form can be written

$$\bar{V} \times \bar{\Omega} = \frac{1}{\rho} \nabla p + \frac{1}{2} \nabla \bar{V} \cdot \bar{V} - \bar{F} \qquad (2)$$

If we take the curl of both sides and use the equation of continuity,

$$\nabla \cdot \bar{V} = 0 \qquad (3)$$

there results

$$\bar{V} \cdot \nabla \bar{\Omega} = \bar{\Omega} \cdot \nabla \bar{V} + \nabla \times \bar{F}$$

The tangential component of this vector equation is

$$u \frac{\partial \eta}{\partial r} + w \frac{\partial \eta}{\partial z} + \frac{v \zeta}{r} = \xi \frac{\partial v}{\partial r} + \zeta \frac{\partial v}{\partial z} + \frac{u \eta}{r} + \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}$$

which, using the definitions of vorticity, can be simplified to

$$\frac{d}{dt}\left(\frac{\eta}{r}\right) = \frac{1}{r}\left[\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right) + \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right] \qquad (5)$$

Since the circulation is directly proportional to the quantity $\frac{\eta}{r}$, this equation expresses the law governing the time change of circulation around the tangential vorticity ring. The first term on the right is the change in the axial direction of the centrifugal force, the second term is the change of the radial blade force in the axial direction, and the third term is the change of the axial blade force in the radial direction. A little thought will show that these terms, in each case, represent moments tending to cause rotation of a particle about a tangential axis. It is in this manner that a non-conservative force field tends to effect a change in the tangential vortex ring.

The time derivative on the left is taken along a streamline and is written

$$\frac{d}{dt}\left(\frac{\eta}{r}\right) = u \frac{\partial}{\partial r}\left(\frac{\eta}{r}\right) + w \frac{\partial}{\partial z}\left(\frac{\eta}{r}\right) \qquad (6)$$

In order to compare the relative magnitude of the terms on the right we use the definition of tangential velocity and write

$$u \frac{\partial}{\partial r}\left(\frac{\eta}{r}\right) = \frac{u}{r}\frac{\partial^2 u}{\partial r \partial z} - \frac{u}{r}\frac{\partial^2 w}{\partial r^2} - \frac{u}{r^2}\frac{\partial u}{\partial z} + \frac{u}{r^2}\frac{\partial w}{\partial r}$$

$$w \frac{\partial}{\partial z}\left(\frac{\eta}{r}\right) = \frac{w}{r}\frac{\partial^2 u}{\partial z^2} - \frac{w}{r}\frac{\partial^2 w}{\partial r \partial z}$$

Remembering that for the axial flow machine the radial velocity is small compared to the axial velocity and expecting from physical considerations that the velocity distribution will be smooth, it appears that

$$u \frac{\partial}{\partial r}\left(\frac{\eta}{r}\right) << w \frac{\partial}{\partial z}\left(\frac{\eta}{r}\right)$$

This inequality will be useful for the first approximations but must be more closely investigated for the final approximations. It is conceivable that for flow that differs greatly from vortex flow the radial change of tangential vorticity would be of such magnitude that this inequality is not justified. It will however be useful to group the smaller terms separately, that is, to write

$$\frac{\partial \eta}{\partial z} = \frac{1}{w}\left[\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right) + \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} - u\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] \tag{7}$$

Using the definition of vorticity we can write an equation for the radial velocity as follows

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} = \frac{1}{w}\left[\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right) + \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} - u\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] \tag{8}$$

so that the left side, if set equal to zero, would be a linear partial differential equation for which a solution is known.

Since the fluid is inviscid the force of the blade on the fluid will act normal to the blade and hence normal to the relative velocity so that

$$u F_r + (v - \omega r) F_\vartheta + w F_z = 0 \tag{9}$$

The radial velocity and the radial force are both small compared to the other force and velocity components so that with very good approximation

$$F_z = - \frac{v - \omega r}{w} F_\vartheta \tag{10}$$

Using this relation in the equations for radial velocity we have one of the final equations of the iteration process

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} = \frac{1}{w}\left[\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right) + \frac{\partial}{\partial r}\left(\frac{v-\omega r}{w} F_\vartheta\right) + \frac{\partial F_r}{\partial z} - u\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] \tag{11}$$

For determining a relation between the tangential velocity and the tangential force we have the second equation of motion

$$\frac{d}{dt}(rv) = u\frac{\partial}{\partial r}(rv) + w\frac{\partial}{\partial z}(rv) = r F_\vartheta$$

which states that the variation along a streamline of the moment of momentum is equal to the moment of the tangential force about the axis of rotation. Then since we will prescribe the tangential force we can express the tangential velocity as an integral along a cylinder.

$$\int \frac{\partial}{\partial z}(rv)\,dz = \int \left(\frac{rF_\theta}{w} - \frac{u}{w}\frac{\partial}{\partial r} rv \right) dz$$

$$v = v_{-\infty} + \int_{\beta=-\infty}^{\beta=z} \left\{ \frac{F_\theta}{w} - \frac{u}{rw}\frac{\partial}{\partial r}(rv) \right\} d\beta \tag{12}$$

The continuity equation, when integrated, provides an equation for the axial velocity in terms of the radial velocity

$$w = w_{-\infty} - \int_{\beta=-\infty}^{\beta=z} \frac{1}{r}\frac{\partial}{\partial r}(ru)\,d\beta \tag{13}$$

We have developed three basic equations, Eqs. 11, 12, 13, to be used in formulating the linearized problem, for the first approximation, and in the construction of an iteration procedure whereby more exact approximations may be obtained.

b) The Linearized Problem

In order to linearize the basic equations derived in the previous section we shall consider axial flow, where the radial velocity is very small compared to the axial velocity,

$$u << w$$

and will assume that the vorticity is transmitted axially downstream with the mean axial velocity w_o so that

$$\frac{d}{dt}\left(\frac{\eta}{r}\right) \doteq w_o \frac{\partial}{\partial z}\left(\frac{\eta}{r}\right)$$

Furthermore, if the radial force is small, we have for calculating the radial velocity the expression

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} = \frac{1}{w_o}\left[\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right) + \frac{\partial}{\partial r}\left(\frac{v - \omega r}{w} F_\vartheta\right)\right] \tag{14}$$

For calculating the linearized tangential velocity Eq. 12 becomes

$$v = v_{-\infty} + \int_{\beta=-\infty}^{\beta=z} \frac{F_\vartheta(r,\beta)}{w_o}\, d\beta \tag{15}$$

and the linearized axial velocity is, from Eq. 13,

$$w = w_o - \int_{\beta=-\infty}^{\beta=z} \frac{1}{r}\frac{\partial}{\partial r}(ru)\, d\beta \tag{16}$$

We are now able to outline a solution to the linearized problem. We will consider here the inverse problem and will specify the two force components F_r and F_ϑ. We will prescribe $F_r << F_\vartheta$ and $F_\vartheta = F_\vartheta(r,z)$, and will specify the boundary conditions applicable to axial flow.

First the tangential velocity is determined by

$$v = v_{-\infty} + \int_{\beta = -\infty}^{\beta = z} \frac{F_{\vartheta}(r, \beta)}{w_0} \, d\beta \tag{15}$$

Knowing the tangential velocity we can determine the radial velocity from Eq. 14

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} = f(r, z)$$

$$= \frac{1}{w_0}\left[\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right) + \frac{v - \omega r}{r w_0} \frac{\partial}{\partial r}(r F_\theta)\right] \tag{14}$$

with the boundary condition that

$$u = 0 \; ; \quad r = r_1, r_2 \; , \quad z = \pm \infty$$

We have here an elliptic partial differential equation for the radial velocity. Since we know the right hand side, the unhomogeneous part, and have complete boundary values we can solve this equation for the radial velocity. This equation was solved by Marble[1] in terms of a Green's function $G(r, z; \alpha, \beta)$ which gives the velocity induced at any point of a circle r,z by a unit change in tangential vorticity of a vortex ring of radius α at an axial coordinate β. By Marble's solution the radial velocity is expressed as

$$u = \int_{-\infty}^{\infty} \int_{r_1}^{r_2} f(\alpha, \beta) \, G(r, z; \alpha, \beta) \, d\alpha \, d\beta \tag{16}$$

where

$$f(\alpha, \beta) = \frac{1}{w_0}\left[\frac{\partial}{\partial r}\left(\frac{v^2}{r}\right) + \frac{v - \omega r}{r w_0} \frac{\partial}{\partial r} r F_\vartheta\right]\Bigg|_{\alpha, \beta} \tag{17}$$

$$G(r, z; \alpha, \beta) = \sum_{n=1}^{\infty} \frac{\alpha \, U_1(\epsilon_n r) \, U_1(\epsilon_n \alpha) \, e^{-\epsilon_n |z - \beta|}}{2 \, \epsilon_n \, \vartheta_n^2} \tag{18}$$

$U_1(\epsilon_n r)$ is the linear combination of Bessel functions of order one

$$U_1(\epsilon_n r) = J_1(\epsilon_n r) Y_1(\epsilon_n r_1) - J_1(\epsilon_n r_1) Y_1(\epsilon_n r) \tag{19}$$

and the characteristic values ϵ_n are chosen for these particular boundary conditions. The quantity ν_n is the norm of $U_1(\epsilon_n r)$ over the interval r_1, r_2.

$$\nu_n^2 = \frac{r_2^2 U_0(\epsilon_n r_2) - r_1^2 U_0(\epsilon_n r_1)}{2} \tag{20}$$

Knowing now the radial and tangential velocities we have from Eq. 16 the linearized axial velocity

$$w = w_0 - \int_{\beta=-\infty}^{\beta=z} \frac{1}{r} \frac{\partial}{\partial r}(ru)\, d\beta \tag{16}$$

We have formulated above the linearized solution as developed by Marble[1]. It should be noted that the Green's function as derived is independent of the manner of linearization provided, of course, that the unhomogeneous part $f(\alpha, \beta)$ is known. If then we have another value of $f(\alpha, \beta)$, more exact than the above linearized value, we can use the same integral (Eq. 16) to determine a more exact radial velocity distribution.

This integral, involving an infinite sum of Bessel functions, is extremely difficult to calculate. A solution for a particular blade loading was obtained by Marble[1]. For the axial flow machine where the boundaries exclude the region near the axis, i.e., the small values of the arguments of the Bessel function, we can use the asymptotic values of the Bessel function with good accuracy. This makes it possible to sum the infinite series so that the Green's function can be expressed in closed form.

c) Asymptotic Value of Green's Function

An asymptotic approximation, in closed form, is desired for the Green's function

$$G(r, z; \alpha, \beta) = \sum_{n=1}^{\infty} \frac{\alpha U_1(\epsilon_n r)\, U_1(\epsilon_n \alpha)\, e^{-\epsilon_n |z-\beta|}}{2\, \epsilon_n \nu_n^2} \tag{18}$$

where

$$\left.\begin{aligned} U_1(\epsilon_n r) &= J_1(\epsilon_n r)\, Y_1(\epsilon_n r_1) - J_1(\epsilon_n r_1)\, Y_1(\epsilon_n r) \\ U_0(\epsilon_n r) &= J_0(\epsilon_n r)\, Y_0(\epsilon_n r_1) - J_0(\epsilon_n r_1)\, Y_0(\epsilon_n r) \end{aligned}\right\} \tag{19}$$

$$\nu_n^2 = \frac{r_2^2\, U_0(\epsilon_n r_2) - r_1^2\, U_0(\epsilon_n r_1)}{2} \tag{20}$$

and ϵ_n is found from the boundary condition that

$$U_1(\epsilon_n r_2) = J_1(\epsilon_n r_2)\, Y_1(\epsilon_n r_1) - J_1(\epsilon_n r_1)\, Y_1(\epsilon_n r_2) = 0 \tag{21}$$

For "large" $(\epsilon_n r)$ asymptotic values of the Bessel functions are found in Jahnke-Emde[6].

$$\begin{aligned} J_1(\epsilon_n r) &\cong \frac{\cos(\epsilon_n r - \frac{3\pi}{4})}{\sqrt{\frac{1}{2}\pi \epsilon_n r}} \\ J_0(\epsilon_n r) &\cong \frac{\cos(\epsilon_n r - \frac{\pi}{4})}{\sqrt{\frac{1}{2}\pi \epsilon_n r}} \\ Y_1(\epsilon_n r) &\cong \frac{\sin(\epsilon_n r - \frac{3\pi}{4})}{\sqrt{\frac{1}{2}\pi \epsilon_n r}} \\ Y_0(\epsilon_n r) &\cong \frac{\sin(\epsilon_n r - \frac{\pi}{4})}{\sqrt{\frac{1}{2}\pi \epsilon_n r}} \end{aligned} \tag{22}$$

The corresponding characteristic values ϵ_n are found from the solution of Eq. 21, with the Bessel functions replaced by their asymptotic values, Eq. 22.

$$\epsilon_n = \frac{n\pi}{r_2 - r_1} \quad ; \quad n = 1, 2, 3, \cdots \tag{23}$$

Using these characteristic values in Eq. 22 we find the complete asymptotic value of the Bessel functions, compatible with the boundary conditions of the problem.

$$\begin{aligned}
J_1(\epsilon_n r) &\simeq \frac{\cos\left(\frac{n\pi r}{r_2 - r_1} - \frac{3\pi}{4}\right)}{\pi\sqrt{\frac{1}{2} n r}} \\
J_0(\epsilon_n r) &\simeq \frac{\cos\left(\frac{n\pi r}{r_2 - r_1} - \frac{\pi}{4}\right)}{\pi\sqrt{\frac{1}{2} n r}} \\
Y_1(\epsilon_n r) &\simeq \frac{\sin\left(\frac{n\pi r}{r_2 - r_1} - \frac{3\pi}{4}\right)}{\pi\sqrt{\frac{1}{2} n r}} \\
Y_0(\epsilon_n r) &\simeq \frac{\sin\left(\frac{n\pi r}{r_2 - r_1} - \frac{\pi}{4}\right)}{\pi\sqrt{\frac{1}{2} n r}}
\end{aligned} \tag{24}$$

The asymptotic value of U_0, U_1, and ν_n follow easily and the asymptotic value of the Green's function can be written as the infinite sum

$$G = \sum_{n=1}^{\infty} \sqrt{\frac{\alpha}{r}} \frac{1}{n\pi} \sin n\pi \frac{r - r_1}{r_2 - r_1} \sin n\pi \frac{\alpha - r_1}{r_2 - r_1} e^{-n\pi \frac{|z-\beta|}{r_2 - r_1}}$$

This series can be summed and expressed in closed form.

$$G(r, z; \alpha, \beta) \simeq \frac{1}{4\pi}\sqrt{\frac{\alpha}{r}} \ln \frac{\cosh \pi \frac{z-\beta}{r_2 - r_1} - \cos \pi \frac{(r - r_1) + (\alpha - r_1)}{r_2 - r_1}}{\cosh \pi \frac{z-\beta}{r_2 - r_1} - \cos \pi \frac{(r - r_1) - (\alpha - r_1)}{r_2 - r_1}} \tag{25}$$

d) The Iteration Procedure

Using the asymptotic value of the Green's function as derived in the previous section three general equations can be written for the iteration procedure.

We will use the velocities resulting from a solution of the linearized problem to obtain the second approximation and will obtain subsequent approximations using velocities of the preceding approximation. Here we are, in effect, assuming that the vorticity is, in each case, transported by the velocities of the preceding approximation.

The radial and tangential force components are prescribed and we start with

$$u_o = 0$$

$$w_o = \text{mean axial velocity}$$

If the velocities which are to be obtained by the nth approximation are denoted by the subscript n the equations for the nth approximation velocities are

$$v_n = v_{-\infty} + \int_{\beta=-\infty}^{\beta=z} \left(\frac{F_\theta(r,\beta)}{w_{n-1}(r,\beta)} - \frac{u_{n-1}(r,\beta)}{r w_{n-1}(r,\beta)} \frac{\partial r v}{\partial r} \right) d\beta \tag{26}$$

$$u_n = \int_{\beta=-\infty}^{\beta=\infty} \int_{\alpha=r_1}^{\alpha=r_2} f_{n-1}(\alpha,\beta)\, G(r,z;\alpha,\beta)\, d\alpha\, d\beta \tag{27}$$

where

$$f_{n-1}(\alpha,\beta) = \frac{1}{w_{n-1}} \left\{ \frac{\partial}{\partial z}\left(\frac{v_n^2}{r}\right) + \frac{\partial}{\partial r}\left(\frac{v_n - \omega r}{w_{n-1}} F_\theta\right) + \frac{\partial F_r}{\partial z} - u_{n-1} \frac{\partial}{\partial r} \frac{1}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) \right\}$$

$$G(r,z;\alpha,\beta) = \frac{1}{4\pi}\sqrt{\frac{\alpha}{r}}\, \ln \frac{\cosh \pi \frac{\beta - z}{r_2 - r_1} - \cos \pi \frac{(r-r_1)+(\alpha - r_1)}{r_2 - r_1}}{\cosh \pi \frac{\beta - z}{r_2 - r_1} - \cos \pi \frac{(r-r_1)-(\alpha - r_1)}{r_2 - r_1}}$$

$$w_n = w_{-\infty} - \int_{\beta=-\infty}^{\beta=z} \frac{1}{r}\frac{\partial}{\partial r}(ru_n)\, d\beta \qquad (28)$$

These three equations, 26, 27, 28, are the basis of the iteration process which is adapted to mechanical calculations in the next section. We will make use of the important fact that $G(r,z;\alpha,\beta)$ is the same for each iteration.

IV. Adaptation of the Iteration Process to Mechanical Calculation Methods

The integrands of the expressions for the tangential and axial velocity components, Eqs. 26 and 28, are regular over the regions considered and can be integrated by any numerical method with relatively little difficulty. Mechanical calculations can be performed using the equations in their present form.

The integration of the equation for the radial velocity Eq. 27 becomes rather involved by any method, and for mechanical calculations the form of the equation must be radically changed. An investigation of the Green's function, $G(r,z;\alpha,\beta)$ will reveal a logarithmic singularity at $\alpha = r$, $\beta = z$. To determine the radial velocity at any point r,z the integration must be carried out over the entire region. The logarithmic singularity of the integration represents the influence at r,z of an axial change of vorticity at r,z.

It will be advantageous to perform the integration in three parts, one part being over the regular region, away from the singularity, and the other two parts being over a small region which includes the singularity. Integration over the latter region containing the singular point consists of evaluating a regular part and evaluating a part containing a purely logarithmic singularity. With this in mind we write the radial velocity as the sum of three integrals.

$$u = u' + u'' + u''' \qquad (29)$$

$$u = \int_{-\infty}^{\infty}\int_{r_1}^{r_2} f(\alpha,\beta)\, G(r,z;\alpha,\beta)\, d\alpha\, d\beta \tag{27}$$

$$u' = \left\{ \int_{-\infty}^{z-\epsilon}\int_{r_1}^{r_2} + \int_{z-\epsilon}^{\infty}\int_{r_1}^{r_2} + \int_{z-\epsilon}^{z+\epsilon}\int_{r_1}^{r-\delta} + \int_{z-\epsilon}^{z+\epsilon}\int_{r+\delta}^{r_2} \right\} f G\, d\alpha\, d\beta \tag{30}$$

$$u'' = \int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} f(\alpha,\beta,\frac{1}{4\pi}\sqrt{\frac{\alpha}{r}}\, \ln\left(\frac{\cosh\pi\frac{\beta-z}{r_2-r_1} - \cos\pi\frac{(r-r_1)+(\alpha-r_1)}{r_2-r_1}}{\dfrac{\cosh\pi\frac{\beta-z}{r_2-r_1} - \cos\pi\frac{(r-r_1)-(\alpha-r_1)}{r_2-r_1}}{\left(\frac{\beta-z}{r_2-r_1}\right)^2 + \left(\frac{\alpha-r}{r_2-r_1}\right)^2}} \right) d\alpha\, d\beta \tag{31}$$

$$u''' = -\int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} f(\alpha,\beta)\frac{1}{4\pi}\sqrt{r}\, \ln\left(\left(\frac{\beta-z}{r_2-r_1}\right)^2 + \left(\frac{\alpha-r}{r_2-r_1}\right)^2\right) d\alpha\, d\beta \tag{32}$$

Obviously the integrand for u' is finite but we need to show that the integrand of u'' is regular. It is easily shown that

$$\lim_{\alpha=r,\,\beta=z} \left(\frac{1}{4\pi}\sqrt{\frac{\alpha}{r}}\, \ln \frac{\cosh\pi\frac{\beta-z}{r_2-r_1} - \cos\pi\frac{(r-r_1)+(\alpha-r_1)}{r_2-r_1}}{\dfrac{\cosh\pi\frac{\beta-z}{r_2-r_1} - \cos\pi\frac{(r-r_1)-(\alpha-r_1)}{r_2-r_1}}{\left(\frac{\beta-z}{r_2-r_1}\right)^2 + \left(\frac{\alpha-r}{r_2-r_1}\right)^2}} \right)$$

$$= \frac{1}{4\pi} \ln\left(\frac{1-\cos 2\pi\frac{r-r_1}{r_2-r_1}}{\pi^2/2} \right) \tag{33}$$

There is a singularity in this integrand, however, but it does not occur near $\alpha, \beta = r, z$. The distance from r,z to the singularity, measured radially, is given by

$$\frac{\alpha - r}{r_2 - r_1} = -2\left(\frac{r - r_1}{r_2 - r_1}\right) \qquad ; \; 0 < \frac{r - r_1}{r_2 - r_1} < \frac{1}{2}$$

$$= 2\left(1 - \frac{r - r_1}{r_2 - r_1}\right) \qquad ; \; \frac{1}{2} < \frac{r - r_1}{r_2 - r_1} < 1$$

From these equations this singularity is seen to lie outside of any region whose center is at r,z. The integration for u'' therefore does not involve a singularity.

The integrand for u''' obviously possesses a purely logarithmic singularity at $\alpha, \beta = r, z$.

a) Evaluation of the Integral for u'

Using the notation that

$$\iint_R = \int_{-\infty}^{z-\epsilon}\int_{r_1}^{r_2} + \int_{z-\epsilon}^{\infty}\int_{r_1}^{r_2} + \int_{z-\epsilon}^{z+\epsilon}\int_{r_1}^{r-\delta} + \int_{z-\epsilon}^{z+\epsilon}\int_{r+\delta}^{r_2} \tag{34}$$

we have

$$u' = \iint_R f(\alpha,\beta)\, G(r,z;\alpha,\beta)\, d\alpha\, d\beta \tag{35}$$

The function f (α,β) will be different for each iteration but the Green's function, $G(r,z;\alpha,\beta)$ will be the same.

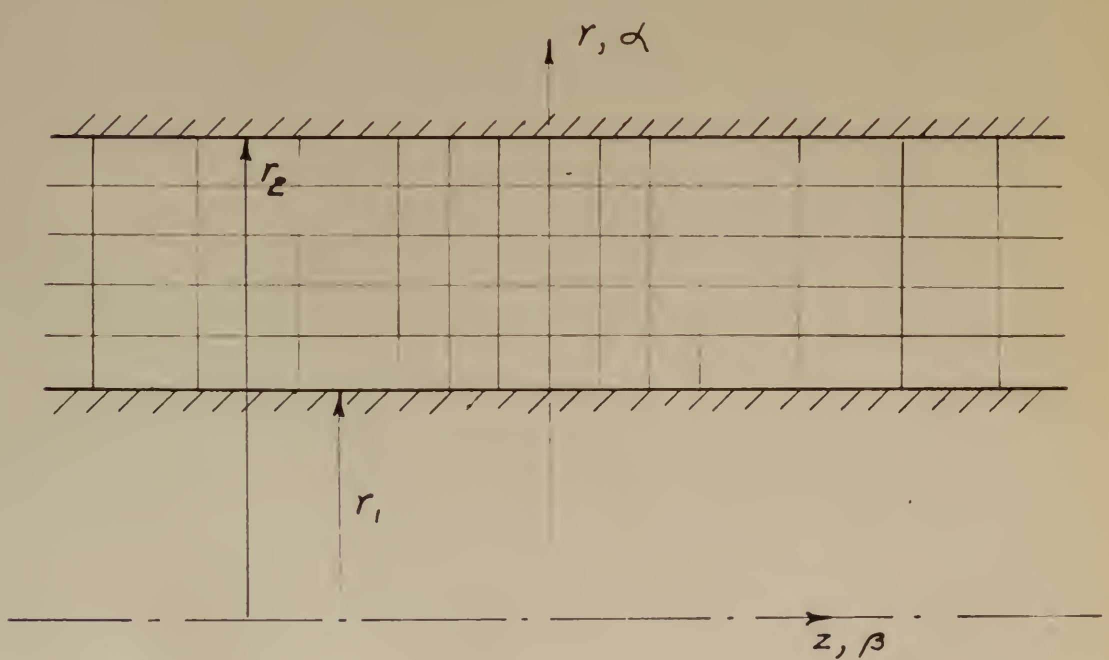

Fig. 2

Subdivision of Flow Field
Into Small Rectangles

If we divide the region into small rectangles as indicated in Fig. 2 we can assume with good accuracy that $f(\alpha, \beta)$ can be represented in one of these sufficiently small regions n by the second degree surface

$$f_n(\alpha, \beta) = C_{n1} + C_{n2}\alpha + C_{n3}\beta + C_{n4}\alpha\beta \tag{36}$$

where four values of $f(\alpha, \beta)$ in the region n are necessary to evaluate the constants C. Suppose the four values of $f(\alpha, \beta)$ are f_{n1} --- f_{n4}. Substituting these values in Eq. 36 we have four equations from which the constants C can be evaluated.

$$\begin{aligned}
f_{n1} &= C_{n1} + C_{n2}\alpha_{n1} + C_{n3}\beta_{n1} + C_{n4}\alpha_{n1}\beta_{n1} \\
f_{n2} &= C_{n1} + C_{n2}\alpha_{n2} + C_{n3}\beta_{n2} + C_{n4}\alpha_{n2}\beta_{n2} \\
f_{n3} &= C_{n1} + C_{n2}\alpha_{n3} + C_{n3}\beta_{n3} + C_{n4}\alpha_{n3}\beta_{n3} \\
f_{n4} &= C_{n1} + C_{n2}\alpha_{n4} + C_{n3}\beta_{n4} + C_{n4}\alpha_{n4}\beta_{n4}
\end{aligned} \tag{37}$$

from which

$$C_{n1} = \frac{\begin{vmatrix} f_{n1} & \alpha_{n1} & \beta_{n1} & \alpha_{n1}\beta_{n1} \\ f_{n2} & \alpha_{n2} & \beta_{n2} & \alpha_{n2}\beta_{n2} \\ f_{n3} & \alpha_{n3} & \beta_{n3} & \alpha_{n3}\beta_{n3} \\ f_{n4} & \alpha_{n4} & \beta_{n4} & \alpha_{n4}\beta_{n4} \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_{n1} & \beta_{n1} & \alpha_{n1}\beta_{n1} \\ 1 & \alpha_{n2} & \beta_{n2} & \alpha_{n2}\beta_{n2} \\ 1 & \alpha_{n3} & \beta_{n3} & \alpha_{n3}\beta_{n3} \\ 1 & \alpha_{n4} & \beta_{n4} & \alpha_{n4}\beta_{n4} \end{vmatrix}} \tag{38}$$

$$C_{n2} = \text{etc.}$$

Four constants C_{n1} --- C_{n4} must be determined for each rectangle. It will be convenient to use $f(\alpha,\beta)$ evaluated at the corners of the rectangle for this. Using the above notation we can write

$$u' = \sum_n \iint_R (C_{n1} + C_{n2}\alpha + C_{n3}\beta + C_{n4}\alpha\beta)\, G(r,z;\alpha,\beta)\, d\alpha\, d\beta \quad (39)$$

and can further define

$$\begin{aligned} I_{n1} &= \iint_R G_n \, d\alpha\, d\beta \\ I_{n2} &= \iint_R \alpha G_n \, d\alpha\, d\beta \\ I_{n3} &= \iint_R \beta G_n \, d\alpha\, d\beta \\ I_{n4} &= \iint_R \alpha\beta\, G_n \, d\alpha\, d\beta \end{aligned} \quad (40$$

so that

$$u' = \sum_n \left(C_{n1} I_{n1} + C_{n2} I_{n2} + C_{n3} I_{n3} + C_{n4} I_{n4} \right) \quad (41)$$

Here the C_{ni} depend only on the function $f(\alpha,\beta)$ and must be determined for each iteration. The I_{ni} depend only on the Green's function and can be determined once and for all. It should be noted however that the I_{ni} are different for each r,z; that is, we must have a certain set of I_{ni} to use in determining the velocity at each point. If we denote by u'^m the velocity at the center of the mth rectangle and use the superscript m to denote the corresponding I_{ni}, then the velocity u' at the center of the mth rectangle is

$$u'^m = \sum_{\substack{n \\ n\neq m}} \left[C_{n1} I^m_{n1} + C_{n2} I^m_{n2} + C_{n3} I^m_{n3} + C_{n4} I^m_{n4} \right] \quad (42)$$

The I_{ni}^{m} are numerous and are quite difficult to calculate. However since the Green's function was derived by considering only the fundamental equations and the boundary condition and hence is completely independent of the prescribed blade loading or the blade shape, these values I_{ni}^{m} can be used for any axial flow problem for which the hub-tip ratio is the same as that for which the I_{ni}^{m} are calculated.

Considerable simplification of the problem of calculating the I_{ni}^{m} is possible because of the symmetry of the Green's function. This function is symmetrical about $\beta = z$, a fact which reduces the number of I_{ni}^{m} by almost one-half. Furthermore it possesses another sort of symmetry in β and z in that the influence of the vorticity at β on the velocity at z is the same as the influence of the vorticity at z on the velocity at β. This condition again reduces the number of I_{ni}^{m} by almost one-half. This latter "reciprocal" relation can be used with the radial distances α and r for further simplification, but the relation is slightly different because of the factor $\sqrt{\frac{\alpha}{r}}$ in the Green's function.

The I_{ni}^{m} will of course depend on the number and size of the rectangles into which the region is divided. The rectangles should be small in the vicinity of the region where the blade forces act and can be larger away from the blades. According to the linearized results given by Marble[1] it appears that in most cases it is not necessary to carry the integration more than about five blade span lengths (r_2-r_1) upstream and downstream of the blades, but this range may not be sufficient for the higher approximations and it is believed advisable to calculate the I_{ni}^{m} for a somewhat greater range.

The expression for u'^m is best expressed in matrix form for mechanical calculations. We define

$$\left[u'^m \right] = \begin{bmatrix} u'^1 \\ u'^2 \\ u'^3 \\ \vdots \\ \\ \end{bmatrix} \tag{43}$$

$$\left[C_{ni} \right] = \begin{bmatrix} C_{1i} \\ C_{2i} \\ C_{3i} \\ \vdots \\ \\ \end{bmatrix} ; \; i = 1,2,3,4 \tag{44}$$

$$\left[I^m_{ni} \right] = \begin{bmatrix} I^1_{1i} & I^1_{2i} & I^1_{3i} & \cdots \\ I^2_{1i} & I^2_{2i} & I^2_{3i} & \cdots \\ I^3_{1i} & \vdots & \vdots & \\ \vdots & & & \\ \end{bmatrix} \tag{45}$$

Using this notation we can write

$$\left[u'^m \right] = \sum_{i=1,2,3,4} \left[C_{ni} \right] \left[I^m_{ni} \right] \tag{46}$$

The four column matrices $\left[C_{ni} \right]$ must be evaluated for each iteration. The four square matrices $\left[I^m_{ni} \right]$ can be determined once and for

all as soon as the subdivision of the field into small rectangles is decided upon.

b) Evaluation of the Integral for u''

The integral u'' is the regular part of the integration over the small region containing the point at which the velocity is to be calculated

$$u'' = \int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} f(\alpha,\beta)\frac{1}{4\pi}\sqrt{\frac{\alpha}{r}}\ \ln\left(\frac{\cosh\pi\frac{\beta-z}{r_2-r_1}-\cos\pi\frac{(r-r_1)+(\alpha-r_1)}{r_2-r_1}}{\frac{\cosh\pi\frac{\beta-z}{r_2-r_1}-\cos\pi\frac{(r-r_1)-(\alpha-r_1)}{r_2-r_1}}{\left(\frac{\beta-z}{r_2-r_1}\right)^2+\left(\frac{\alpha-r}{r_2-r_1}\right)^2}}\right)d\alpha\, d\beta \qquad (31)$$

We have from Eq. 33 the value of the integrand at the center point of the region, i.e., at $\alpha,\beta = r, z$, as

$$f(\alpha,\beta)\frac{1}{4\pi}\ln\frac{1-\cos 2\pi\frac{r-r_1}{r_2-r_1}}{\pi^2/2} \qquad (47)$$

It can be shown that this point is a saddle point of the logarithmic function in the integrand since it represents a maximum with respect to α and a minimum with respect to β.

It follows then that if the region is sufficiently small and if the function $f(\alpha,\beta)$ is smooth we can use the value of the logarithmic term at the center as an approximate mean value (constant) so that an approximate value for u'' is

$$u'' = \frac{1}{4\pi}\ln\frac{1-\cos 2\pi\frac{r-r_1}{r_2-r_1}}{\pi^2/2}\int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} f(\alpha,\beta)\, d\alpha\, d\beta \qquad (48)$$

Since this integral u'' represents the effect of only one small region the above expression is sufficiently accurate unless the change of vorticity in this region is much greater than anywhere else in the flow field.

We can write the integral above for the mth rectangle as

$$u''^{m} = J^{m} D^{m} \tag{49}$$

where

$$J^{m} = \frac{1}{4\pi} \ln \frac{1 - \cos 2\pi \frac{r - r_1}{r_2 - r_1}}{\pi^2/2} \tag{50}$$

is evaluated at the mth point and

$$D^{m} = \int_{z-\epsilon}^{z+\epsilon} \int_{r-\delta}^{r+\delta} f(\alpha, \beta)\, d\alpha\, d\beta \tag{51}$$

is taken over the mth rectangle.

The J^{m} are determined once and for all as soon as the subdivisions of the field are decided upon. The D^{m} must be determined for each iteration.

c) Evaluation of the Integral for u'''

The integral u''' is the singular part of the integration over the small region containing the point at which the velocity is being calculated.

$$u''' = -\int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} f(\alpha,\beta)\frac{1}{4\pi}\sqrt{\frac{\alpha}{r}}\, \ln\left(\left(\frac{\beta-z}{r_2-r_1}\right)^2+\left(\frac{\alpha-r}{r_2-r_1}\right)^2\right) d\alpha\, d\beta \qquad (32)$$

We define a new function momentarily as

$$g(\alpha,\beta;r) = f(\alpha,\beta)\frac{1}{4\pi}\sqrt{\frac{\alpha}{r}} \qquad (52)$$

and write the integral as

$$u''' = -\int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} g(\alpha,\beta,r)\, \ln\left(\left(\frac{\beta-z}{r_2-r_1}\right)^2+\left(\frac{\alpha-r}{r_2-r_1}\right)^2\right) d\alpha\, d\beta \qquad (53)$$

The function $g(\alpha,\beta,r)$ is regular and possesses higher order derivative so that it can be expanded about the center of the rectangle in a Maclaurin's series in the two variables α, β.

$$\begin{aligned} g(\alpha,\beta,r) = {} & g(r,z) + \left(\frac{\partial g}{\partial\alpha}\right)_{r,z}(\alpha-r) + \left(\frac{\partial g}{\partial\beta}\right)_{r,z}(\beta-z) \\ & + \frac{1}{2!}\left[\left(\frac{\partial^2 g}{\partial\alpha^2}\right)_{r,z}(\alpha-r)^2 + 2\left(\frac{\partial^2 g}{\partial\alpha\partial\beta}\right)_{r,z}(\alpha-r)(\beta-z) + \right. \\ & \left. \left(\frac{\partial^2 g}{\partial\beta^2}\right)_{r,z}(\beta-z)^2\right] + \cdots \end{aligned} \qquad (54)$$

If we substitute this expansion of $g(\alpha,\beta)$ in the integral u'' and neglect terms of fourth and higher orders we have

$$u''' = g(r,z)\,K_1 + \frac{1}{2}\left(\frac{\partial^2 g}{\partial\alpha^2}\right)_{r,z} K_2 + \frac{1}{2}\left(\frac{\partial^2 g}{\partial\beta^2}\right)_{r,z} K_3 \qquad (55)$$

where

$$K_1 = -\int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} \ln\left(\left(\frac{\beta-z}{r_2-r_1}\right)^2+\left(\frac{r-\alpha}{r_2-r_1}\right)^2\right) d\alpha\, d\beta$$

$$K_2 = -\int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} (\alpha-r)^2 \ln\left(\left(\frac{\beta-z}{r_2-r_1}\right)^2+\left(\frac{r-\alpha}{r_2-r_1}\right)^2\right) d\alpha\, d\beta$$

$$K_3 = -\int_{z-\epsilon}^{z+\epsilon}\int_{r-\delta}^{r+\delta} (\beta-z)^2 \ln\left(\left(\frac{\beta-z}{r_2-r_1}\right)^2+\left(\frac{r-\alpha}{r_2-r_1}\right)^2\right) d\alpha\, d\beta$$

The integrals involving the odd powers of $(\alpha - r)$ or $(\beta - z)$ vanish because of the symmetry of the logarithmic term. The above integrals can be evaluated by straightforward integration by parts to give

$$K_1 = -\left[\delta\epsilon \ln\frac{\epsilon^2+\delta^2}{(r_2-r_1)^2} - 12\,\delta\epsilon + 4\,\delta^2 \tan^{-1}\frac{\epsilon}{\delta} + 4\epsilon^2 \tan^{-1}\frac{\delta}{\epsilon}\right] \qquad (56)$$

$$K_2 = -\frac{1}{3}\Big[4\,\delta^3\epsilon \ln\frac{\epsilon^2+\delta^2}{(r_2-r_1)^2} + \delta\epsilon\left(2\epsilon^2 - \frac{26}{3}\delta^2\right) + 6\,\delta^4 \tan^{-1}\frac{\epsilon}{\delta} - 2\,\epsilon^4 \tan^{-1}\frac{\delta}{\epsilon}\Big] \qquad (57)$$

$$K_3 = -\frac{1}{3}\Big[4\,\delta\epsilon^3 \ln\frac{\epsilon^2+\delta^2}{(r_2-r_1)^2} + \delta\epsilon\left(2\delta^2 - \frac{26}{3}\epsilon^2\right) + 6\,\epsilon^4 \tan^{-1}\frac{\delta}{\epsilon} - 2\,\delta^4 \tan^{-1}\frac{\epsilon}{\delta}\Big] \qquad (58)$$

The three constants, K_1, K_2, K_3, are determined once and for all as soon as the subdivisions of the field are decided upon.

By straightforward differentiation we can express the derivatives of $g(\alpha, \beta, r)$ in terms of $f(\alpha, \beta)$ and we can write the integral u''' for the velocity at the rectangle m as

$$u'''^{m} = E_1^m K_1^m + E_2^m K_2^m + E_3^m K_3^m \tag{59}$$

where K_1, K_2, K_3 are defined above and where

$$E_1^m = (g)_{r,z} = \frac{1}{4\pi} f(r,z)$$

$$E_2^m = \frac{1}{2}\left(\frac{\partial^2 g}{\partial \alpha^2}\right)_{r,z} = -\frac{1}{32\pi r^2} f(r,z) + \frac{1}{8\pi r}\left(\frac{\partial f}{\partial \alpha}\right)_{r,z} + \frac{1}{8\pi}\left(\frac{\partial^2 f}{\partial \alpha^2}\right)_{r,z}$$

$$E_3^m = \frac{1}{2}\left(\frac{\partial^2 g}{\partial \beta^2}\right)_{r,z} = \frac{1}{8\pi}\left(\frac{\partial^2 f}{\partial \beta^2}\right)_{r,z}$$

d) The Complete Integral for u

We have evaluated the integral for the radial velocity in three parts as given by Eqs. 42, 49, and 59. Combining these results we can write a complete expression for the radial velocity at the center r,z of the rectangular subdivision m.

$$u^m = \sum_{\substack{n \\ n \neq m}} \left[C_{n1} I_{n1}^m + C_{n2} I_{n2}^m + C_{n3} I_{n3}^m + C_{n4} I_{n4}^m \right] + D^m J^m + \sum_{i=1,2,3} E_i^m K_i^m \qquad (60)$$

The constants C, D and E are evaluated from the unhomogeneous part $f(\alpha, \beta)$ of the differential equation for the radial velocity (Eqs. 11 and 27) and are to be determined in the manner described in the preceding sections. These constants must be determined for each iteration.

The constants I, J and K depend only on the Green's function, (i.e., the boundary conditions), the hub-tip ratio, and the size of the subdivisions and can therefore be evaluated once and for all. They are used in the same manner in each iteration step.

V. DISCUSSION

A method of solving the hydrodynamic equations for the incompressible flow of an inviscid fluid through an axial flow turbomachine has been developed as an iteration procedure. The next logical step of this approach to the problem is the evaluation of the invariant terms of the iteration expression for the radial velocity. Once this is done for the several likely hub-tip ratios it will be relatively easy to calculate, by mechanical means, solutions for any axial flow machine with any prescribed blade loading.

In determining the degree of accuracy that will be required the restrictions (incompressibility, etc.) which were imposed to simplify the mathematical problem must be considered and evaluated. The solution should be consistent with these restrictions and of sufficient accuracy to indicate the proper trends of the variables. It is not certain that the linearized solution meets these requirements in all cases. The iteration procedure can be used to determine the accuracy of the linearized solution and if necessary to obtain solutions of greater accuracy.

VI APPENDIX

a) A Brief Comparison of the Axial Flow and Mixed Flow Problems

The extension of the more recent axial flow solutions to apply to mixed flow machines would be extremely difficult and would require consideration of several points, not significant in axial flow, which are of utmost importance in mixed flow. Three essential differences between the two problems are pointed out here in order to indicate which of the assumptions used in the axial flow analysis would not be applicable to a mixed flow analysis.

(1) The mixed flow machine contains continuous vanes as contrasted with the rotor and stator blade rows in an axial flow machine. The vanes therefore cannot be "twisted" without introducing excessive tilt away from radial so that the centrifugal forces cause large bending moments in the vane and prevent operation of the machine at extremely high speeds. A geometric relation must hold between the relative velocity, the vane forces on the fluid and the shape of the vanes throughout the region where the vanes are present, whereas for blade rows as in axial flow this relation holds only in the region of the narrow blade and, in fact, may be concentrated in a "lifting line" for a good approximation.

(2) The larger radial velocities which naturally occur in mixed flow prohibit the simplifying assumption used

for axial flow that the radial velocity is very small compared to the axial velocity.

(3) The boundary conditions of the mathematical problem of the mixed flow compressor are much more complicated than those for axial flow and greatly increase the difficulty of obtaining a solution. The variables are not separable in this case. It should be noted too that the boundaries will probably be very different for each mixed flow problem whereas they are always essentially the same for axial flow.

Several types of vanes are possible but two special vane shapes are likely to be of interest.

The first of these might be called "radial vanes". These vanes are generated by radial lines through the axis of rotation. Here the radial force is zero. Radial vanes are necessary in a high speed machine because of the high centrifugal forces. If radial velocities and (or) pitch angles are large as in mixed flow then the angles between the vanes and the hub or shroud will be acute, thus increasing boundary layer effects.

The second type might be called "normal vanes". These vanes are generated by lines through the axis but tilted in a meridional plane so as to be normal to the meridional trace of the streamline at all points. Here the vane force has two components, one acting along the meridional trace of the streamsurface and one acting tangentially. This vane is not structurally adequate for extremely high speed rotors

but does have the good feature that angles between the vane and the hub or shroud are right angles, a fact which may minimize boundary layer effects. It is interesting to note that one force component accelerates the fluid in its path in a meridional plane and the other component accelerates it tangentially so that it appears that no "wasted" forces are present. For the axial flow machine radial vanes and normal vanes are about the same.

The selection of either radial or normal vanes will lead to great simplification of the mathematical problem.

b) Some Consequences of the Hydrodynamic Equations as Applied to Turbomachines

The Euler equations of motion are written in vector form and in cylindrical component form for the isentropic flow of an inviscid and incompressible fluid acted upon by non-conservative body forces. The flow is symmetrical about the axis.

$$\bar{V} \cdot \nabla \bar{V} = -\frac{1}{\rho} \nabla p + \bar{F} \tag{1}$$

$$\begin{aligned} u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r \\ \frac{u}{r}\frac{\partial rv}{\partial r} + w\frac{\partial v}{\partial z} &= F_\vartheta \\ u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z \end{aligned} \tag{2}$$

By use of a vector identity these equations may be written in terms of the vorticity

$$\bar{V} \times \bar{\Omega} = \nabla\left(\frac{P}{\rho}\right) - \bar{F} \tag{3}$$

where, since the flow is isentropic and adiabatic,

$$\nabla\left(\frac{P}{\rho}\right) = \frac{1}{\rho}\nabla p + \frac{1}{2}\bar{V}\cdot\bar{V}$$

is the gradient of the total energy of the fluid. In component form Eq. (3) is

$$\begin{aligned} v\zeta - w\eta &= \frac{\partial}{\partial r}\left(\frac{P}{\rho}\right) - F_r \\ w\xi - u\zeta &= -F_\vartheta \\ u\eta - v\xi &= \frac{\partial}{\partial z}\left(\frac{P}{\rho}\right) - F_z \end{aligned} \tag{4}$$

Several interesting relations between velocity, vorticity, total energy, vane shape, etc., can be derived from these equations if we consider the flow between two concentric bounding surfaces acted upon by vanes or blades rotating with an angular velocity ω. We will assume an infinite number of vanes (or blades) so that the forces of the vane on the fluid may be represented by body forces symmetrical about the axis. Furthermore since the fluid is inviscid these forces are normal to the vane and hence to the relative velocity so that

$$(\bar{V} - \bar{\omega r}) \cdot \bar{F} = 0 \tag{5}$$

If we take the scaler product of the velocity and both sides of Eq. 3 these results

$$\bar{V} \cdot \nabla\left(\frac{P}{\rho}\right) = \bar{V} \cdot \bar{F} = \cancelto{0}{(\bar{V} - \bar{\omega r}) \cdot \bar{F}} + \omega r F_\theta$$

or

$$\frac{d}{dt}\left(\frac{P}{\rho}\right) = \omega r F_\theta \tag{6}$$

From the second equation of motion

$$\frac{d}{dt}(rv) = r F_\theta \tag{7}$$

From Eqs. 6 and 7 we see that

$$\frac{d}{dt}\left(\frac{P}{\rho}\right) = \omega \frac{d}{dt}(rv) \quad \text{, where rotating vanes act}$$

$$= 0 \quad \text{, where no rotating vanes act}$$

If we consider the case where the fluid has constant energy far upstream and is free of vorticity far upstream and make the stipulation that the total energy of the fluid is changed only by the action of the moving vanes (i.e., adiabatic flow) then Eq. 8 can be integrated along each streamsurface with the result that

$$\left(\frac{P}{\rho}\right)_{r,z} - \left(\frac{P}{\rho}\right)_{\text{far upstream}} = (\omega r v)_{r,z} - (\omega r v)_{\text{far upstream}}$$

For this case of uniform energy and no vorticity far upstream, $\frac{P}{\rho}$ and $\omega r v$ are constant far upstream so that:

$$\nabla\left(\frac{P}{\rho}\right) = 0 \qquad \text{; upstream of the vanes} \tag{9}$$

$$= \nabla(\omega r v) \text{ ; in regions of and downstream of the vanes if no forces such as stationary vane forces occur}$$

If any forces, such as those resulting from stationary vanes or stators, have tangential components then we can only say that $\frac{d}{dt}\left(\frac{P}{\rho}\right) = 0$ along the streamsurface since in this case $\omega \frac{d}{dt}(rv) = 0$, but $\frac{d}{dt}(rv) \neq 0$

Combining Eqs. 3 and 9 we have

$$\bar{V} \times \bar{\Omega} = \nabla(\omega r v) - \bar{F}$$

and if we multiply both sides by the vorticity

$$\bar{\Omega} \cdot \bar{V} \times \bar{\Omega} = \bar{\Omega} \cdot \nabla(\omega r v) - \bar{\Omega} \cdot \bar{F} \equiv 0$$

we see that

$$\bar{\Omega} \cdot \left[\nabla(\omega r v) - \bar{F}\right] = 0 \tag{10}$$

Using only the definition of vorticity it is easily shown that

$$\bar{\Omega} \cdot \nabla(\omega r v) = 0 \tag{11}$$

and hence

$$\bar{\Omega} \cdot \nabla\left(\frac{p}{\rho}\right) = 0 \tag{12}$$

and further from Eq. 10

$$\bar{\Omega} \cdot \bar{F} \quad = 0 \tag{13}$$

Using the definition of vorticity and Eq. 9 it is seen that

$$\omega \bar{r} \times \bar{\Omega} = \nabla\left(\frac{P}{\rho}\right) \tag{14}$$

Combining Eq. 14 and Eq. 3

$$\left(\bar{V} - \omega \bar{r}\right) \times \bar{\Omega} = -\bar{F} \tag{15}$$

For this special case we can draw the following interesting conclusions:

Upstream of the Vanes:

1. The vorticity is zero.
2. The total energy of the fluid is uniform.

In the Region Where Vane Forces Act:

1. The vorticity vector is tangent to the vane surface. (Eqs. 5 and 13).
2. The vorticity vector is tangent to a surface of constant total energy. (Eq. 12)

3. The bound vorticity is tangent to the line of intersection of a vane and a concentric surface of constant total energy.

Downstream of the Vanes:

1. The vorticity vector is tangent to the relative velocity vector (relative to the rotating rotor).
2. The vorticity vector is tangent to a surface of constant total energy, this surface being the streamsurface.

REFERENCES

1. Marble, Frank E., "The Flow of a Perfect Fluid Through an Axial Turbomachine with Prescribed Blade Loading"; Journal of the Aeronautical Sciences, Vol. 15, No. 8, pp. 473-485, August, 1948.

2. Meyer, Richard, "Beitrag zur Theorie feststehender Schaufelgitter"; Leeman (Zurich), 1946. (Also available as British A.R.C. Trans., A.R.C. 8869, F.M. 830, T.J.R.74).

3. Traupel, Walter, "New General Theory of Multistage Axial Flow Turbomachines", Nav Ships 250-445-1, Navy Dept. (Trans. by C. W. Smith, General Electric Corporation).

4. von Karman, Th., and Burgess, J. M., "General Aerodynamics Theory-Perfect Fluids"; Aerodynamic Theory, Vol. II, pp. 102-214, Durand Reprinting Committee, California Institute of Technology.

5. Wislicenus, George F., "Fluid Mechanics of Turbomachinery"; McGraw-Hill Book Company, Inc., 1947.

6. Jahnke, E., and Emde, F., "Funktionentafeln, pp. 137-138; Teubner, Leipzig, 1933.